Nomenclature

c_p	specific heat capacity
Pr	Prandtl number
Q	rate of heat transfer
q_w	wall heat flux
Re_τ	Reynolds number = $u_\tau \delta / \nu$
T	temperature
t	time
T_m	bulk mean temperature
T_τ	friction temperature = $q_w / \rho c_p u_\tau$
u_i, u, v, w	velocity component
u_τ	friction velocity = $\sqrt{\tau_w / \rho}$
x_1, x	streamwise direction
x_2, y	wall-normal direction
x_3, z	spanwise direction
Brackets	
$\overline{(\ )}$	statistically averaged
$\langle\ \rangle$	averaged over channel section
Greek	
α	thermal diffusivity
δ	channel half width
μ	dynamic viscosity
ν	kinematic viscosity
ρ	density
τ_w	wall shear stress
θ	transformed temperature
Superscipts	
$(\ )^*$	normalized by δ
$(\ )^+$	normalized by u_τ, ν and T_τ

We begin with the non-dimensional form of the energy equation:

$$\frac{\partial T^+}{\partial t^*} + u_j \frac{\partial T^+}{\partial x_j^*} = \frac{1}{Re_\tau Pr} \frac{\partial^2 T^+}{\partial x_j^{*2}} \tag{1}$$

The dimensionless temperature $T^+(x^*, y^*, z^*)$ can be broken down into a mean temperature which increases linearly up the channel (with x^*) and a fluctuating component $\theta^+(x^*, y^*, z^*)$ such that:

$$T^+ = \frac{d\langle \bar{T}_m^+ \rangle}{dx^*} x^* - \theta^+ \tag{2}$$

The mean dimensionless temperature is defined as:

$$\langle \bar{T}_m^+ \rangle = \int_0^1 \bar{u}_1^+ \bar{T}^+ dy^* \bigg/ \int_0^1 \bar{u}_1^+ dy^* \tag{3}$$

$\langle \bar{T}_m^+ \rangle$ is averaged over time (statistically) and cross section and is thus only a function of x^*.

The derivative may be found as follows:

$$\begin{aligned}
Q &= \dot{m} c_p dT \\
2 q_w \delta z dx &= 2\rho \langle \bar{u} \rangle c_p dT \delta \delta z \\
\frac{dT}{dx} &= \frac{q_w}{\rho c_p \delta} \\
\frac{\rho u_\tau c_p \delta}{q_w} \frac{dT}{dx} &= \frac{u_\tau}{\langle \bar{u} \rangle} \\
\frac{d(\rho u_\tau c_p T / q_w)}{d(x/\delta)} &= \frac{1}{\langle \bar{u}^+ \rangle} \\
\frac{d\langle \bar{T}_m^+ \rangle}{dx^*} &= \frac{1}{\langle \bar{u}^+ \rangle}
\end{aligned} \tag{4}$$

Using these transformations the energy equation can be found:

$$\frac{\partial}{\partial t^*}\left(\frac{d\langle \bar{T}_m^+ \rangle}{dx^*} x^* - \theta^+\right) + u_j^+ \frac{\partial}{\partial x_j^*}\left(\frac{d\langle \bar{T}_m^+ \rangle}{dx^*} x^* - \theta^+\right) = \frac{1}{Re_\tau Pr} \frac{\partial^2}{\partial x_j^{*2}}\left(\frac{d\langle \bar{T}_m^+ \rangle}{dx^*} x^* - \theta^+\right) \tag{5}$$

In the first left hand term in (5) the first part of the derivative relates to how the mean non-dimensional temperature gradient varies with time, since it is statistically averaged it does not vary with time and thus the term is equal to zero. In the second term on the left side the mean non-dimensional temperature gradient is a function of only x^* and thus when differentiated with respect to y^*, z^* the derivative is zero, however for x^* it is non-zero, whereas the fluctating part is a function of all three directions. The second derivative of the mean non-dimensional temperature gradient is zero if the gradient is assumed to be linear (as was the case in the formation of (4)). Thus (5) may be rewritten as:

$$\begin{aligned}
-\frac{\partial \theta^+}{\partial t^*} + u_1^+ \frac{\partial}{\partial x^*}\left(\frac{d\langle \bar{T}_m^+ \rangle}{dx^*} x^*\right) - u_j^+ \frac{\partial \theta^+}{\partial x_j^*} &= -\frac{1}{Re_\tau Pr} \frac{\partial^2 \theta^+}{\partial x_j^{*2}} \\
\frac{\partial \theta^+}{\partial t^*} + u_j^+ \frac{\partial \theta^+}{\partial x_j^*} &= \frac{1}{Re_\tau Pr} \frac{\partial^2 \theta^+}{\partial x_j^{*2}} + u_1^+ \frac{d\langle \bar{T}_m^+ \rangle}{dx^*} \\
\frac{\partial \theta^+}{\partial t^*} + u_j^+ \frac{\partial \theta^+}{\partial x_j^*} &= \frac{1}{Re_\tau Pr} \frac{\partial^2 \theta^+}{\partial x_j^{*2}} + \frac{u_1^+}{\langle \bar{u}^+ \rangle}
\end{aligned} \tag{6}$$

(6) may be solved directly by some programs, however for use in Code Saturne further manipulation is required since it operates on the generic transport equation which utilises $f(t, x, y, z)$ rather than dimensionless quantities $f(t^*, x_i^*)$.

The following relations are used:

$$\begin{aligned}
t^* &= t u_\tau / \delta \\
x_i^* &= x_i / \delta \\
Re_\tau &= u_\tau \delta / nu \\
Pr &= \mu c_p / k \\
\mu &= \rho \nu \\
\alpha &= \rho c_p / k
\end{aligned}$$

Applying these relations to (6) yields:

$$
\begin{aligned}
\frac{\partial\theta^+}{\partial(tu_\tau\delta)} + u_j^+\frac{\partial\theta^+}{\partial(x_j/\delta)} &= \frac{\nu k}{u_\tau c_p\mu\delta}\frac{\partial^2\theta^+}{\partial(x_j/\delta)^2} + \frac{u_1^+}{\langle\bar{u}^+\rangle} \\
\frac{\delta}{u_\tau}\frac{\partial\theta^+}{\partial t} + \delta u_j^+\frac{\partial\theta^+}{\partial x_j} &= \frac{\delta k}{\rho u_\tau c_p}\frac{\partial^2\theta^+}{\partial x_j^2} + \frac{u_1^+}{\langle\bar{u}^+\rangle} \\
\frac{\partial\theta^+}{\partial t} + u_\tau u_j^+\frac{\partial\theta^+}{\partial x_j} &= \frac{1}{\alpha}\frac{\partial^2\theta^+}{\partial x_j^2} + \frac{u_\tau}{\delta}\frac{u_1^+}{\langle\bar{u}^+\rangle} \\
\frac{\partial\theta^+}{\partial t} + u_j\frac{\partial\theta^+}{\partial x_j} &= \frac{1}{\alpha}\frac{\partial^2\theta^+}{\partial x_j^2} + \frac{1}{\delta}\frac{u_1}{\langle\bar{u}^+\rangle} \\
\frac{\partial(\delta\theta^+)}{\partial t} + u_j\frac{\partial(\delta\theta^+)}{\partial x_j} &= \frac{1}{\alpha}\frac{\partial^2(\delta\theta^+)}{\partial x_j^2} + \frac{u_1}{\langle\bar{u}^+\rangle} \qquad (7)
\end{aligned}
$$

$$ \qquad (8) $$

(7) is then the transport equation for a property $\delta\theta^+$ with the additional source term $\frac{u_1}{\langle\bar{u}^+\rangle}$